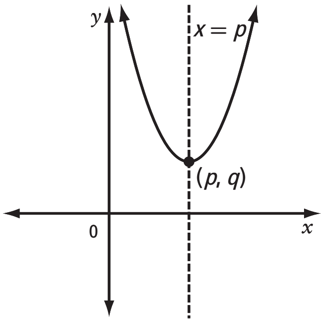
Chapter 3 Quadratic Functions

3.1 Investigating Quadratic Functions in Vertex Form

KEY IDEAS

- For a quadratic function in vertex form, $f(x) = a(x - p)^2 + q$, $a \neq 0$, the graph
  - has the shape of a parabola
  - has its vertex at (p, q)
  - has an axis of symmetry with equation $x = p$
  - is congruent to $f(x) = ax^2$ translated horizontally by p units and vertically by q units



- You can sketch the graph of $f(x) = a(x - p)^2 + q$ by transforming the graph of $f(x) = x^2$.

The parameter *a* gives the direction of opening and the vertical stretch factor.	
$a > 0$	The parabola opens upward.
$a < 0$	The parabola is reflected in the x-axis; it opens downward.
$-1 < a < 1$	The parabola is compressed vertically; it is wider than the graph of $f(x) = x^2$ or $f(x) = -x^2$.
$a > 1$ or $a < -1$	The parabola is stretched vertically; it is narrower than the graph of $f(x) = x^2$ or $f(x) = -x^2$.
The parameter *q* gives the vertical translation.	
$q > 0$	The parabola is translated q units up.
$q < 0$	The parabola is translated q units down.
The parameter *p* gives the horizontal translation.	
$p > 0$	The parabola is translated p units to the right.
$p < 0$	The parabola is translated p units to the left.

Working Example 1: Sketch the Graph of a Quadratic in Vertex Form

State the vertex, the direction of opening, the equation of the axis of symmetry, the domain and range, and the maximum or minimum value of the graph of $y = -3(x + 4)^2 + 1$. Then, sketch the graph.

Solution

Use the values of a, p, and q to determine some characteristics of the function and sketch the graph.

$$y = -3(x + 4)^2 + 1$$

$a \quad p \quad q$

Which of the parameters a, p, and q determine the coordinates of the vertex?

The vertex is (________, ________).

Since $a < 0$, the graph opens ______________.
(*upward* or *downward*)

The equation of the axis of symmetry is ________.

Which of the parameters a, p, and q determine the equation of the axis of symmetry?

The domain of the quadratic is $\{x \mid x \in \mathrm{R}\}$.

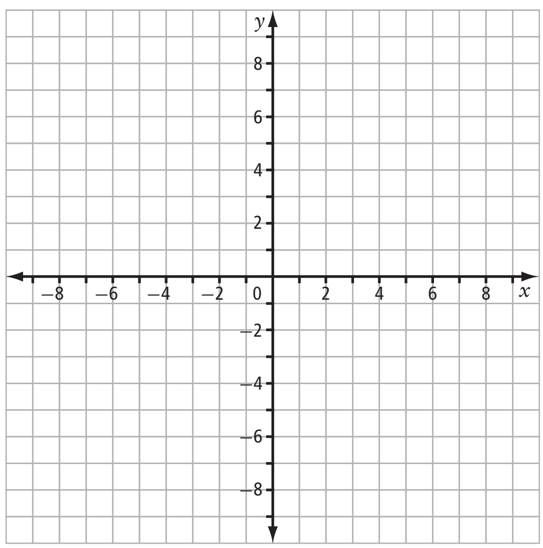
How can you use the value of q to help you determine the range?

The range is $\{y \mid y \leq$ ________, $y \in \mathrm{R}\}$.

The ______________ value is ________.
(*maximum* or *minimum*)

Determine the coordinates of a point (other than the vertex) on the graph.
Choose an x-coordinate and determine the corresponding y-coordinate.
Choose $x = -3$. Substitute this value of x into the function to determine y.

$y = -3(x + 4)^2 + 1$
$y = -3(\mathbf{-3} + 4)^2 + 1$
$y = -3(1)^2 + 1$
$y = -2$



So, one point on the graph is (________, ________).

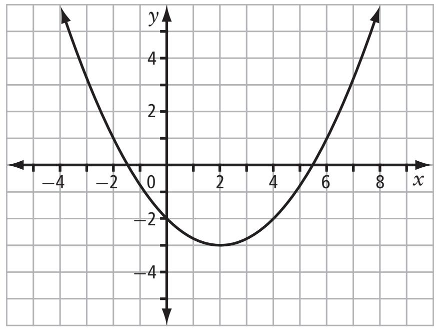
Since every parabola is symmetric about its axis of symmetry, this means that another point on the graph is (________, −2).

Plot the vertex and these two other points to sketch the graph.

Compare this method with the two methods shown on pages 148–149 of *Pre-Calculus 11*.

Working Example 2: Determine a Quadratic Function in Vertex Form Given Its Graph

Determine a quadratic function, in vertex form, for the graph.



Solution

The vertex of the graph is ________.

So, $p =$ ________ and $q =$ ________.

The equation of the function in the form

$y = a(x - p)^2 + q$ is ________________.

Substitute the coordinates of any point from the graph, except the vertex, into the function to determine the value of a.
Use (0, −2):

$$\begin{aligned} y &= a(x - 2)^2 - 3 \\ \mathbf{-2} &= a(\mathbf{0} - 2)^2 - 3 \\ -2 &= 4a - 3 \\ 1 &= 4a \\ \frac{1}{4} &= a \end{aligned}$$

Substitute the parameters a, p, and q to write the equation of the function.

$y =$ ________________

Working Example 3: Determine the Number of *x*-Intercepts Using *a* and *q*

Determine the number of x-intercepts for the quadratic function $f(x) = 4(x - 3)^2$.

Solution

Use the value of a to determine if the graph opens upward or downward.
Use the value of q to determine if the vertex is above, below, or on the x-axis.

a ________ 0 The graph opens ________________.
(< or >) (*upward* or *downward*)

q ________ 0 The vertex is ________________ the x-axis.
(< or = or >) (*on* or *below* or *above*)

How many x-intercept(s) does the function have? ________

How many times does the graph touch the x-axis?

See pages 153–154 of *Pre-Calculus 11* for more examples.

Check Your Understanding

Practise

1. State the coordinates of the vertex and the number of x-intercepts for each of the following functions.

a) $y = (x - 3)^2 + 5$

$p =$ ________ $q =$ ________

vertex: (________, ________)

a ________ 0; the graph opens ________________
(< or >) (*upward* or *downward*)

q ________ 0; there are ________ x-intercepts.
(< or >)

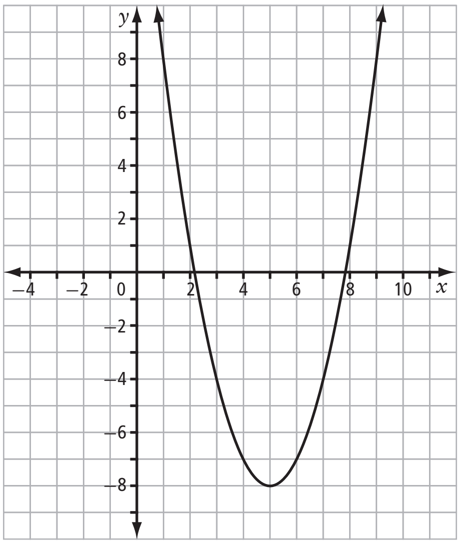
b) $y = -4x^2 + 1$

c) $y = \frac{2}{3}(x - 11)^2$

d) $y = -\left(x + \frac{1}{2}\right)^2 + \frac{7}{3}$

2. State the direction of opening, the equation of the axis of symmetry, and the maximum or minimum value for each of the following.

a)



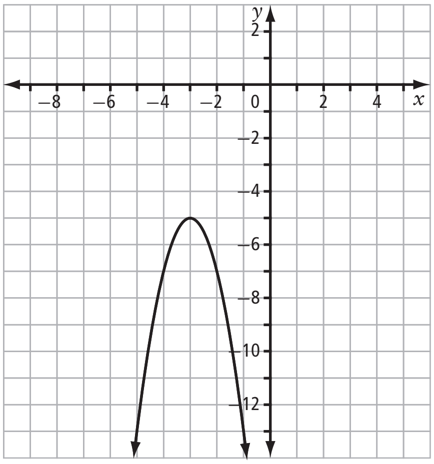
$p =$ ________ $q =$ ________

The graph opens ________________.
(*upward* or *downward*)

The equation of the axis of symmetry is ________.

The ________________ value is ________.
(*maximum* or *minimum*)

b)



$p =$ ________ $q =$ ________

The graph opens ________________.
(*upward* or *downward*)

The equation of the axis of symmetry is ________.

The ________________ value is ________.
(*maximum* or *minimum*)

3. Describe how to obtain the graph of each function from the graph of $y = x^2$. State the domain and the range for each. Then, graph the function.

a) $y = (x + 4)^2 - 2$

$a =$ ________ $p =$ ________ $q =$ ________

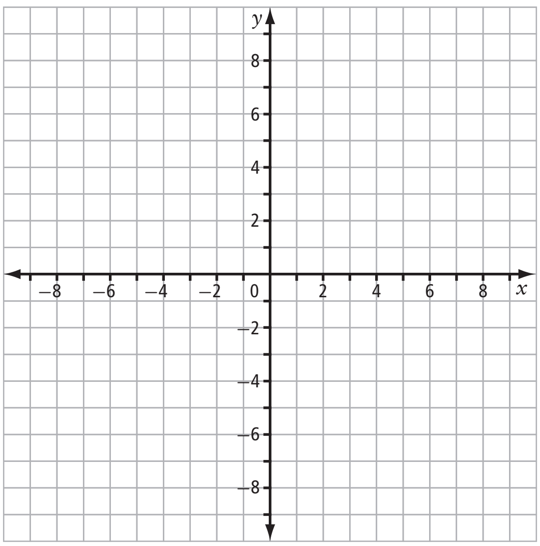
The graph opens ________________.
(*upward* or *downward*)

The ________________ value is ________.
(*maximum* or *minimum*)

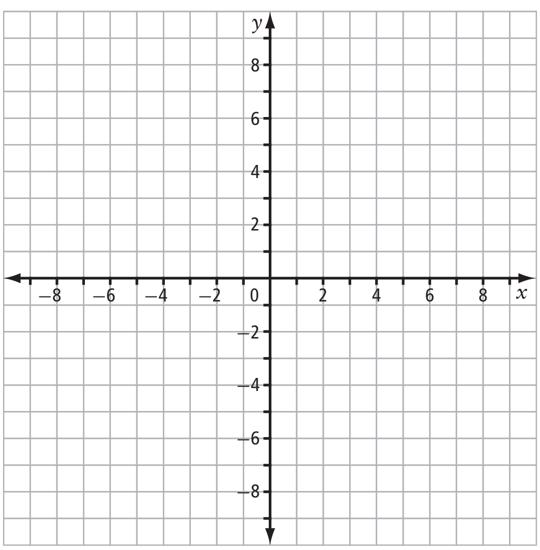
The graph of $y = x^2$ is translated ________ units to the ________________ and
(*left* or *right*)

________ units ________________.
(*up* or *down*)

The domain is ________________. The range is ________________.

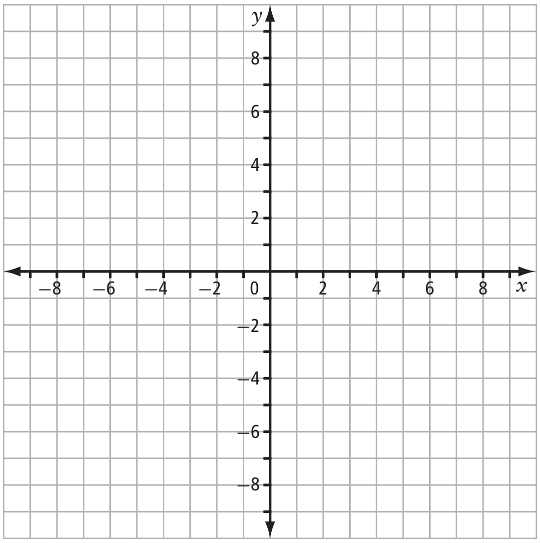


b) $y = -4(x + 7)^2 + 2$

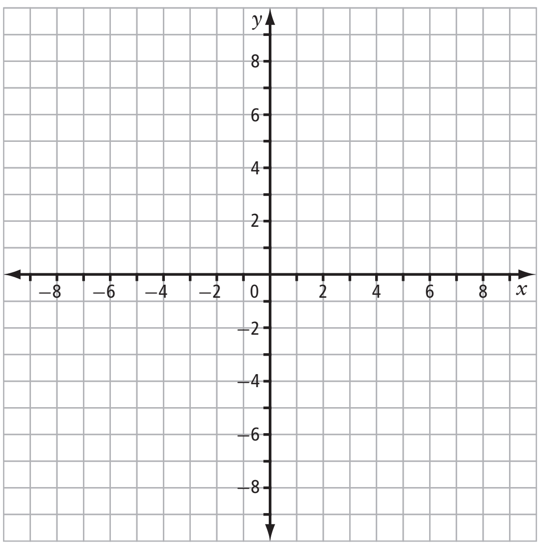


4. Sketch each of the following. Label the vertex and axis of symmetry. State the domain and range.

a) $y = -2(x + 5)^2 + 4$

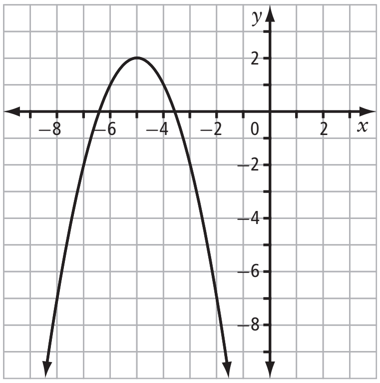


b) $y = \frac{1}{2}(x - 3)^2 - 4$



5. Determine the quadratic function in vertex form for each parabola.

a)



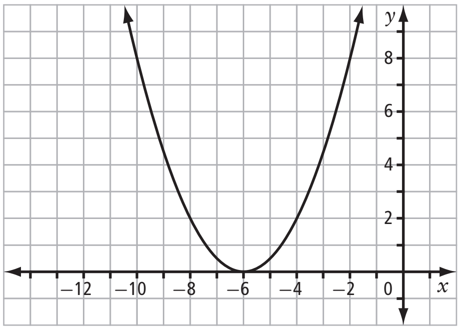
$p =$ ________

$q =$ ________

$a =$ ________

function: ________________

b)



$p =$ ________

$q =$ ________

$a =$ ________

function: ________________

6. Write the quadratic function in vertex form that has the given characteristics.

a) vertex at (0, 4), congruent to $y = 5x^2$

$a =$ ________ $p =$ ________ $q =$ ________

function: ________________

b) vertex at (3, 0), passing through (4, −2)

Substitute the coordinates of the vertex and the point (4, −2) into $y = a(x - p)^2 + q$ to determine a.

c) vertex (1, −1), with y-intercept 3

Substitute the coordinates of the vertex and the coordinates of the y-intercept into $y = a(x - p)^2 + q$ to determine a.

Apply

7. Determine the corresponding point on the transformed graph for the point (−1, 1) on the graph of $y = x^2$.

a) $y = x^2$ is transformed to $y = (x + 5)^2 - 1$.

For $y = (x + 5)^2 - 1$, $p =$ ________ and $q =$ ________.

Apply the horizontal translation of 5 units to the left and the vertical translation of 1 unit down to the point (−1, 1): (−1 + ________, 1 + ________)

The corresponding point of (−1, 1) after the graph is transformed is (________, ________).

This question should help you complete #10 on page 158 of *Pre-Calculus 11*.

b) $y = x^2$ is transformed to $y = 2(x - 2)^2 - 3$.

For $y = 2(x - 2)^2 - 3$, $a =$ ________, $p =$ ________, and $q =$ ________.

Apply the multiplication of the y-values by a factor of 2 to the point (−1, 1):

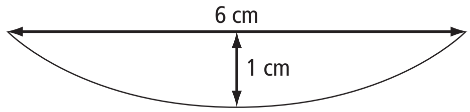
(−1, ________ (1))

Then, apply the horizontal translation of 2 units to the right and the vertical translation of 3 units down to the point (−1, 2): (−1 + ________, 2 + ________)

The corresponding point of (−1, 1) after the graph is transformed is (________, ________).

c) $y = x^2$ is transformed to $y = \frac{1}{2}(x + 1)^2 + 4$.

8. Parabolic mirrors are often used in lights because they give a focused beam. Suppose a parabolic mirror is 6 cm wide and 1 cm deep, as shown.



a) Suppose the vertex of the mirror is at the origin. Determine the quadratic function in vertex form that describes the shape of the mirror.

The coordinates of the vertex are ________.

The coordinates of one endpoint of the mirror are ________.

Use the coordinates of the vertex and the endpoint to determine a.

If the vertex is at the origin, the function is ________________.

b) Now suppose the origin is at the left outer edge of the mirror. Determine the quadratic function in vertex form that describes the mirror.

The coordinates of the vertex are ________.

If the origin is at the left endpoint, the function is ________________.

c) Compare your functions in parts a) and b). How are they similar? How are they different?

9. The points (0, 0) and (8, 0) are on a parabola.

a) How many different parabolas do you think pass through these points?

b) Choose a point to be the maximum for a parabola passing through (0, 0) and (8, 0). Determine the quadratic function in vertex form with this maximum.

c) Choose a different point to be the maximum. Determine the quadratic function of the resulting parabola.

d) How many of the parameters in a quadratic function in vertex form change when you change the location of the vertex? Explain.

Question 15 on page 159 of *Pre-Calculus 11* uses the same concepts as #8 and #9.

Connect

10. The horizontal distance, d, in metres, that a projectile travels is given by $d = -\frac{v^2 \cos \theta \sin \theta}{4.9}$, where v is the initial speed in metres per second and θ is the angle at which the object leaves the ground.

a) Choose an angle, substitute into the formula, and simplify. List the characteristics of the resulting quadratic function.

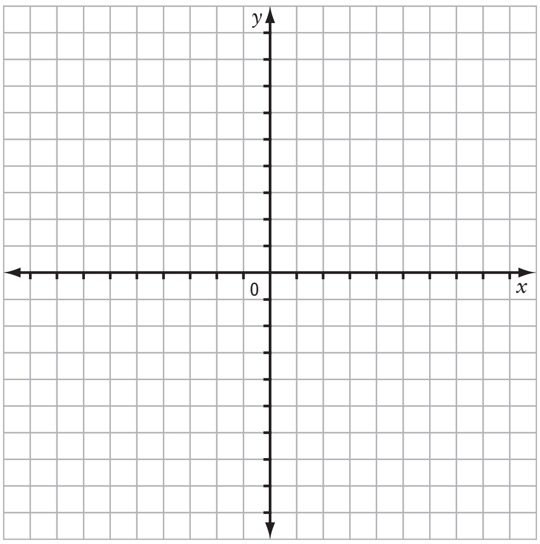
b) Choose a different angle and repeat your work in part a). How do your two resulting quadratic functions compare?

c) List the characteristics that all quadratics resulting from this formula will have in common. Which characteristics will vary?

11. Student council sells school T-shirts for \$12 each. If they raise prices, they hope to make more money. If x represents the price increase and y represents revenue, then the quadratic function $y = -10(x - 9)^2 + 4510$ models this situation.

a) If the price does not change ($x = 0$), what is the current revenue?

b) Sketch a graph of this function.



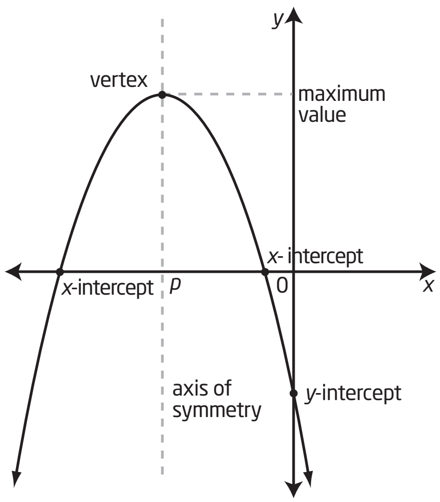
c) Explain why it is reasonable for a quadratic to represent this situation.

d) Determine the price for T-shirts that gives the maximum revenue for the student council. What is this maximum revenue? Justify your answer.

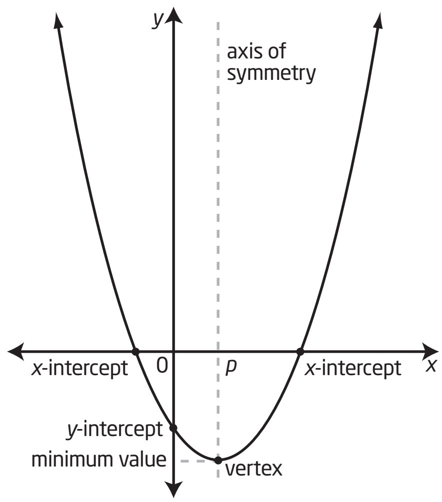
3.2 Investigating Quadratic Functions in Standard Form

KEY IDEAS

- The standard form of a quadratic function is $f(x) = ax^2 + bx + c$ or $y = ax^2 + bx + c$, where $a \neq 0$.
- The graph of a quadratic function is a parabola that
  - is symmetric about a vertical line, called the axis of symmetry, that passes through the vertex
  - opens upward and has a minimum value if $a > 0$
  - opens downward and has a maximum value if $a < 0$
  - has a y-intercept at $(0, c)$ that has a value of c
- Use the graph of a quadratic function to determine the vertex, domain and range, direction of opening, axis of symmetry, x-intercepts, y-intercept, and maximum or minimum value.



Domain: all real numbers
Range: all real numbers less than or equal to the maximum value of y



Domain: all real numbers
Range: all real numbers greater than or equal to the minimum value of y

- For any quadratic function in standard form, the x-coordinate of the vertex is $x = -\frac{b}{2a}$.

See page 166 of *Pre-Calculus 11* to see how to derive the formula for x-coordinate of the vertex.

- For quadratic functions in applied situations,
  - the y-intercept represents the value of the function when the independent variable is 0
  - the x-intercept(s) represent(s) the value(s) of the independent variable when the function is 0
  - the vertex represents the point at which the function reaches its maximum or minimum
  - the domain and range may need to be restricted based on the possible values in the situation

Working Example 1: Characteristics of a Quadratic Function in Standard Form

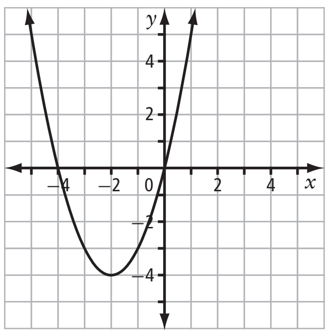
For each graph of a quadratic function given, identify the following:

- the direction of opening
- the coordinates of the vertex
- the maximum or minimum value
- the equation of the axis of symmetry
- the x-intercepts and y-intercept
- the domain and range

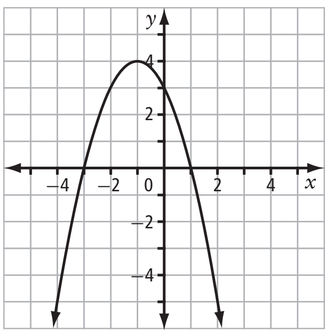
How can you use the x-coordinate of the vertex to determine the y-coordinate of the vertex?

What is the relationship between the x-coordinate of the vertex and the axis of symmetry?

a) $f(x) = x^2 + 4x$



b) $f(x) = -x^2 - 2x + 3$



Solution

a) The graph opens ________________.

The x-coordinate of the vertex is ________.

The y-coordinate of the vertex is ________.

The minimum value of the function is

________.

The equation of the axis of symmetry is

$x =$ ________.

The factored form of the function is $f(x) = x(x + 4)$, so the x-intercepts

are ________ and ________.

The y-intercept is 0 because

____________________________.

The domain is $\{x \mid x \in \mathrm{R}\}$.

The range is $\{y \mid y \geq -4, y \in \mathrm{R}\}$

because ____________________________.

b) The graph opens ________________.

The x-coordinate of the vertex is ________.

The y-coordinate of the vertex is ________.

The maximum value of the function is

________.

The equation of the axis of symmetry is

$x =$ ________.

The factored form of the function is

$f(x) =$ ________________, so the

x-intercepts are ________ and ________.

The y-intercept is ________

because ____________________________.

The domain is ________________.

The range is ________________

because ____________________________.

After completing this example, you should be able to explain the solutions to Example 1 on pages 167–168 of *Pre-Calculus 11*.

Working Example 2: Using a Quadratic Function to Model a Situation

A student council sells memberships for \$6 per year and has 700 members. To increase revenue, they decide to increase the membership cost. The results of a survey indicates that 50 fewer students will buy a membership for every \$1 increase in the membership cost.

a) Write a quadratic function in standard form to model this situation.

b) Graph the quadratic function. What does the shape of the graph communicate about the situation?

c) What are the coordinates of the vertex? What information does it give the student council?

d) Determine if there are any x-intercepts that are relevant. What do these intercepts, if they exist, represent in the situation?

e) What domain and range are logical for this situation? Explain.

Solution

a) Let x represent the number of price increases and $r(x)$ represent the revenue after a given price increase. Then,

revenue = (price)(number of members)

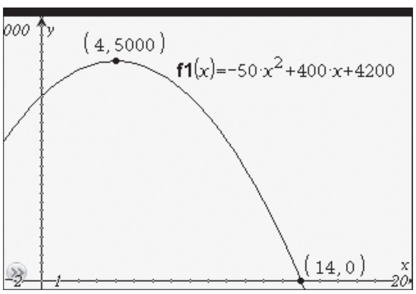
$r(x) = (6 + x)(700 - 50x)$

$r(x) = 4200 - 300x +$ ________

$r(x) =$ ________________

The new price is \$6 plus the number of price increases times \$1, or 6 + 1x. The new number of members is 700 minus the number of price increases times 50, or 700 − 50x.

b) Enter the function on a graphing calculator.



Think about the maximum number of price increases possible and revenue amounts to help you determine window settings.

The shape of the quadratic confirms that if the council raises the membership price, revenue at first increases, but then decreases.

c) The vertex of the graph is located at (4, 5000). This means that a price increase of ________, to a new price of ________, will give the maximum revenue of \$5000.

d) The x-intercept of ________ indicates that if the price is increased by \$ ________, the student council will have no revenue. So, there will be no sales at this price.

e) The domain for this situation is ________________, as 0 is the smallest possible price increase and 14 is the greatest possible price increase.

The range is ________________, as revenue is positive and the maximum revenue is \$5000.

See pages 168–172 of *Pre-Calculus 11* for more examples.

Check Your Understanding

Practise

1. Which of the following functions are quadratic? Rewrite the quadratic functions in standard form.

a) $f(x) = -5(x + 2)^2 + 8$

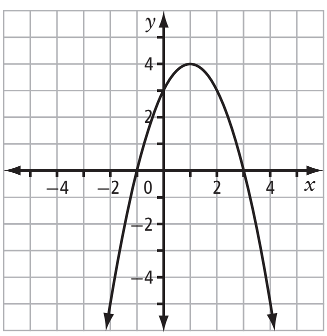
b) $f(x) = 3x^2 + 4^x + 1$

c) $f(x) = (8x + 11)(x - 5)$

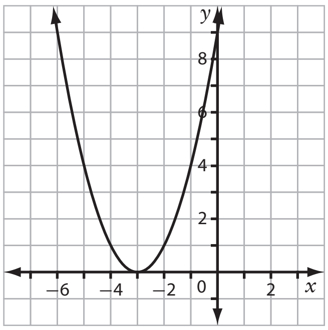
d) $f(x) = (3x - 7)(x - 4)(x + 1)$

2. For each graph, identify the following:
- the coordinates of the vertex
- the equation of the axis of symmetry
- the x-intercepts and y-intercept
- the maximum or minimum value and how it relates to the direction of opening
- the domain and range

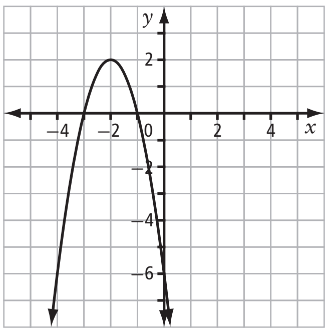
a)



- The vertex is (________, ________).
- The x-coordinate of the vertex gives the equation of the axis of symmetry as $x =$ ________.
- The graph has a y-intercept of ________, and the x-intercepts are ________ and ________.
- Since the graph opens downward, it has a ________________, and that value is given by the y-coordinate of the vertex.
- The domain is ________________. Using the vertex, the range is ________________.

b) 

- vertex: ________
- axis of symmetry: ________
- *y*-intercept: ________
- *x*-intercept: ________
- maximum/minimum value: ________
- domain: ________
- range: ________

c) 

- vertex: ________
- axis of symmetry: ________
- *y*-intercept: ________
- *x*-intercept: ________
- maximum/minimum value: ________
- domain: ________
- range: ________

3. Use technology to graph each of the following functions. Identify the vertex, the axis of symmetry, the maximum or minimum value, the domain and range, and all intercepts. Round values to the nearest tenth, if necessary.

a) $y = x^2 - 5x - 1$

- The vertex is (________, ________).
- The *x*-coordinate of the vertex gives the axis of symmetry as $x =$ ________.
- Since the graph opens ________________ (*upward* or *downward*), it has a ________________ (*maximum* or *minimum*) value and that value is given by the *y*-coordinate of the vertex, which is ________.
- The domain is ________________, and using the vertex, the range is ________________.
- The *x*-intercepts are ________ and ________, and the *y*-intercept is ________.

b) $y = -2x^2 + x + 3$

- vertex: ________
- axis of symmetry: ________
- *y*-intercept: ________
- *x*-intercept: ________
- maximum/minimum value: ________
- domain: ________
- range: ________

c) $y = 0.25x^2 - 5.5x + 27.25$

- vertex: ________
- axis of symmetry: ________
- y-intercept: ________
- x-intercept: ________
- maximum/minimum value: ________
- domain: ________
- range: ________

4. Determine the vertex of each quadratic function.

> The x-coordinate of the vertex is given by $x = -\frac{b}{2a}$. Find the y-coordinate by substituting that value of x into the function.

a) $y = x^2 - 4x - 12$

b) $y = 3x^2 + 6x - 1$

c) $y = -x^2 + 8x + 25$

d) $y = 2x^2 - 6x - 5$

5. Determine the number of x-intercepts for each function. Explain how you know.

a) $y = x^2 + 2x - 35$

The vertex of this function is at (________, ________).

The parabola opens ________________.

There are ________ x-intercepts.

b) a quadratic function with a maximum value at its y-intercept of 12

c) a quadratic function with its vertex located on the x-axis

d) a quadratic function with an axis of symmetry of $x = 2$ and passing through the point (5, 0)

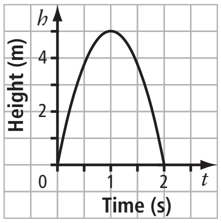
e) a quadratic function with vertex (−2, 5) and passing through the point (1, 18)

f) a quadratic function with a range of $y \leq -7$

See #8 on page 175 of *Pre-Calculus 11* for more practice with this concept.

Apply

6. The graph approximates the height of a soccer ball kicked by the goalkeeper. Use the graph to answer the following questions. Explain what property of the graph led to your answer.



a) From what height is the ball kicked?

b) What is the maximum height of the ball? When does it occur?

c) How long is the ball in the air?

d) What are the domain and range for this situation?

7. When determining the maximum allowable speed for a curve in a road, engineers use the equation $a = \frac{v^2}{r}$, where a is the acceleration, in metres per second squared, experienced by the vehicle as it turns, v is the road speed of the vehicle in metres per second, and r is the radius of the curve, in metres.

a) Suppose that a particular curve has a radius of 25 m. Write the equation for the acceleration of vehicles around this curve.

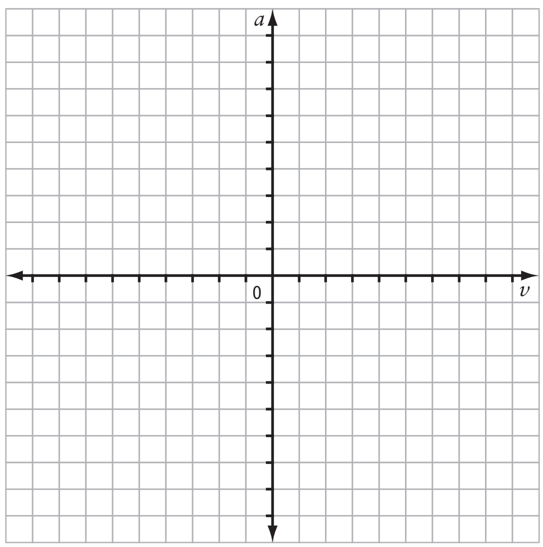
b) Identify the vertex, the equation of the axis of symmetry, and the intercepts of the function. Sketch the graph of the function.

vertex: ________

axis of symmetry: ________

x-intercept: ________

y-intercept: ________



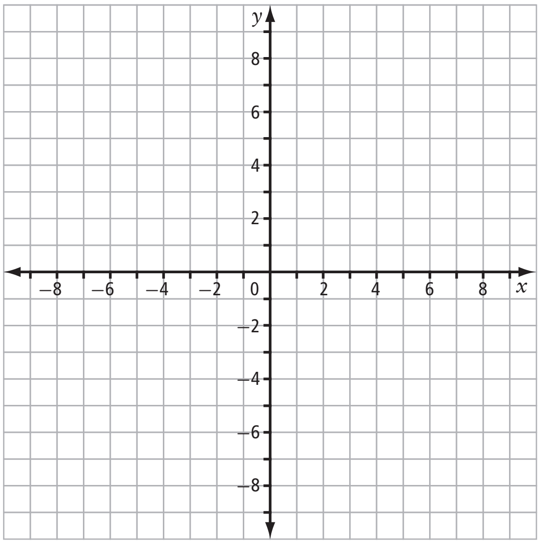
c) Determine the domain of the function. Explain your answer.

d) Determine the range of the function. Explain your answer.

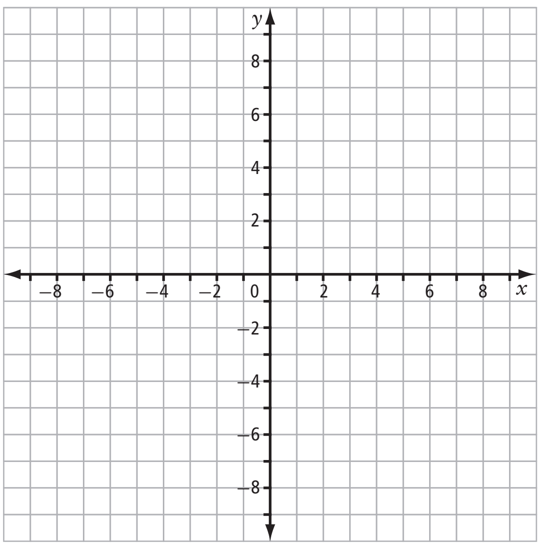
e) The speed limit on the road is equivalent to 14 m/s. The maximum acceleration the engineer wants vehicles to experience on the curve is 6 m/s^2. Does the curve fit the criterion? Explain your answer.

8. Sketch the graph of a quadratic function that has the characteristics given. Is it possible to have more than one correct answer?

a) axis of symmetry $x = 3$ and x-intercepts of 0 and 6



b) x-intercepts of −2 and 3 and range $y \geq -6.25$

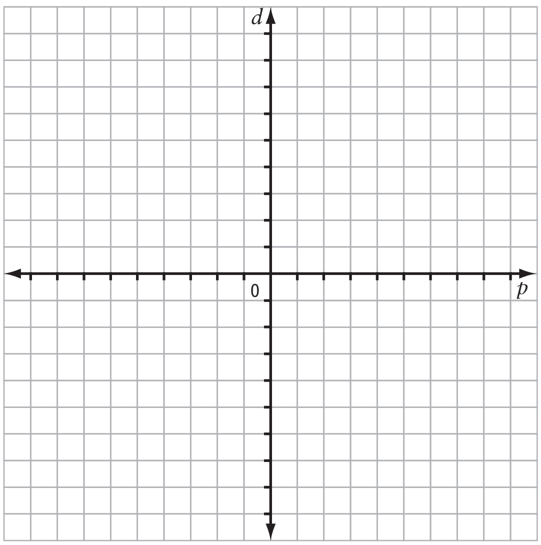


See #10 on page 175 of *Pre-Calculus 11* for a similar question.

9. A manufacturer has determined that demand, d, for its product is given by $d = -p^2 + 24p + 56$, where p is the price of the product, in dollars.

a) Determine the coordinates of the vertex of this function.

b) Sketch the graph of the function.



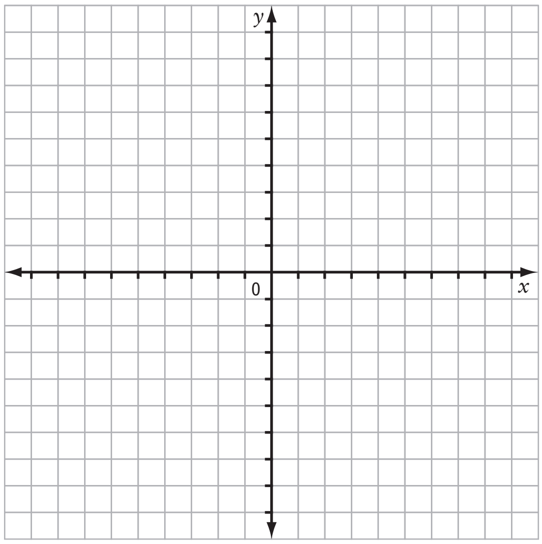
c) What does the vertex represent in this situation?

d) What does the shape of the graph tell the manufacturer in this situation?

10. Brooklyn has 24 m of fencing. She wants to build a rectangular enclosure for her dog with the maximum possible area.

a) Write a function to represent the rectangular area of the enclosure. How do you know that the function is quadratic?

b) Sketch a graph of the function.



c) Determine the coordinates of the vertex. What do these coordinates represent?

d) What are the dimensions of the enclosure that achieve Brooklyn's goals? What area will be available to her dog?

Your work on this question should help you answer #15 and #17 on page 177 of *Pre-Calculus 11*.

Connect

11. a) Explain how knowing the vertex and direction of opening allows you to determine the number of x-intercepts of a quadratic function.

b) Does knowing the equation of the axis of symmetry and direction of opening allow you to determine the number of x-intercepts? Explain using examples.

12. For a quadratic function in standard form $y = ax^2 + bx + c$, the x-coordinate of the vertex is given by $x = -\frac{b}{2a}$. Obtain an expression for the y-coordinate of the vertex by substituting this value into the quadratic. Explain whether you think it is useful to have this expression.

3.3 Completing the Square

KEY IDEAS

- To convert a quadratic function from standard form to vertex form, use an algebraic process called completing the square.

$y = 5x^2 - 30x + 7$	← Standard form
$y = 5(x^2 - 6x) + 7$	Group the first two terms. Factor out the leading coefficient if $a \neq 1$.
$y = 5(x^2 - 6x + \mathbf{9} - \mathbf{9}) + 7$	Add and then subtract the square of half of the coefficient of the x-term to create a perfect square trinomial.
$y = 5[(x^2 - 6x + 9) - 9] + 7$	Group the perfect square trinomial.
$y = 5[(x - 3)^2 - 9] + 7$	Rewrite using the square of a binomial.
$y = 5(x - 3)^2 - 45 + 7$	Simplify.
$y = 5(x - 3)^2 - 38$	← Vertex form

- Converting a quadratic function to vertex form, $y = a(x - p)^2 + q$, reveals the coordinates of the vertex, (p, q).
- You can use information derived from the vertex form to solve problems such as those involving maximum and minimum values.

Working Example 1: Convert from Standard Form to Vertex Form

Rewrite each function in vertex form by completing the square. State the vertex for each.

a) $y = x^2 - 8x + 13$

b) $y = -2x^2 + 12x + 2$

c) $y = 3x^2 + 2x - 1$

Solution

a) Group the first two terms.

$y = (x^2 - 8x) + 13$

Then, add and subtract the square of half the coefficient of the x-term to create a perfect square trinomial.

$y = (x^2 - 8x + 16 - 16) + 13$

$\left(\frac{-8}{2}\right)^2 = (-4)^2$
$= 16$

Factor the first three terms, which will always be a perfect square trinomial.

$y = (x - 4)^2 - 16 + 13$

Simplify.

$y = (x - 4)^2 - 3$

The vertex is at (4, −3).

b) After grouping the first two terms, factor out the coefficient −2 from the group.

$y = -2(x^2 - 6x) + 2$

Complete the square as in part a).

$y = -2(x^2 - 6x + \_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_) + 2$

$y = -2[(x - \_\_\_\_\_\_\_\_)^2 - \_\_\_\_\_\_\_\_] + 2$

$y = -2(x - \_\_\_\_\_\_\_\_)^2 + 18 + 2$

$y = -2(x - \_\_\_\_\_\_\_\_)^2 + \_\_\_\_\_\_\_\_$

Determine the quantity to be added and subtracted by calculating the square of half the coefficient of the x-term.

Remember that the distributive property applies to the fourth term in the parentheses.

The vertex is at (3, 20).

c) Though 2 is not a factor of 3, you can still begin by grouping and factoring.

$y = 3\left(x^2 + \frac{2}{3}x\right) - 1$

$y = 3\left(x^2 + \frac{2}{3}x + \_\_\_\_\_\_\_\_ - \_\_\_\_\_\_\_\_\right) - 1$

$y = 3[(x + \_\_\_\_\_\_\_\_)^2 - \_\_\_\_\_\_\_\_] - 1$

$y = 3(x + \_\_\_\_\_\_\_\_)^2 - \_\_\_\_\_\_\_\_ - 1$

$y = 3(x + \_\_\_\_\_\_\_\_)^2 - \_\_\_\_\_\_\_\_$

The vertex is at ________________.

See pages 184–186 of *Pre-Calculus 11* for similar examples.

Working Example 2: Convert to Vertex Form and Verify

a) Convert the function $y = -3x^2 - 24x - 19$ to vertex form.

b) Verify that the two forms are equivalent.

Solution

a) Complete the square for the function $y = -3x^2 - 24x - 19$.

$y = (\_\_\_\_\_\_ - \_\_\_\_\_\_) - \_\_\_\_\_\_$

$y = -3(\_\_\_\_\_\_ + \_\_\_\_\_\_) - \_\_\_\_\_\_$

$y = -3[(x + \_\_\_\_\_\_)^2 - \_\_\_\_\_\_] - \_\_\_\_\_\_$

$y = -3(x + \_\_\_\_\_\_)^2 + \_\_\_\_\_\_ - \_\_\_\_\_\_$

$y = \_\_\_\_\_\_\_\_\_\_\_\_$

b) Method 1: Use Algebra

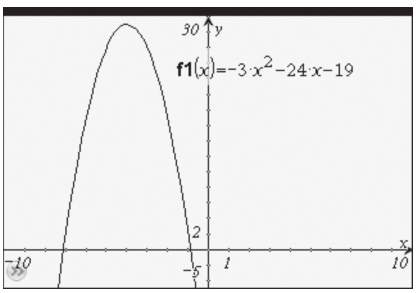
To verify that the two forms are equivalent, expand and simplify the vertex form of the function.

$y = -3(x + 4)^2 + 29$
$y = -3(x + 4)(x + 4) + 29$
$y = -3(x^2 + 4x + 4x + 16) + 29$
$y = -3(x^2 + 8x + 16) + 29$
$y = -3x^2 - 24x - 48 + 29$
$y = -3x^2 - 24x - 19$

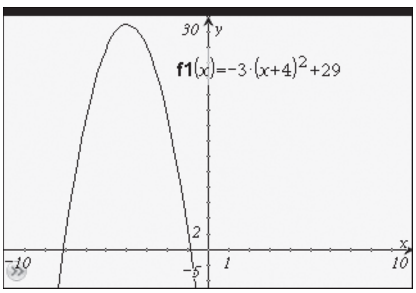
Method 2: Use Technology

Use graphing technology to graph each function.

$y = -3x^2 - 24x - 19$



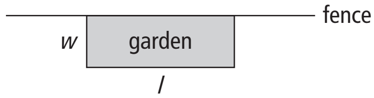
$y = -3(x + 4)^2 + 29$



Since the graphs appear identical, the two forms are equivalent.

Working Example 3: Optimization

Rylee has 12 m of edging material to place along the three sides of the garden to separate it from her lawn. What dimensions will give the maximum area for the garden?



Solution

Let w represent the width of the garden and l represent the length of the garden.

The expression $\underline{\qquad\qquad} + \underline{\qquad\qquad} = 12$ models the edging material available.

Isolate l: $l = 12 - \underline{\qquad\qquad}$.

The area of the garden is given by $A = lw$. Substitute the expression above for l.

$A = (\underline{\qquad\qquad})w$
$A = 12w - 2w^2$

Rearrange the area formula and complete the square.
$A = -2w^2 + 12w$

$A = -2(\underline{\qquad\qquad} - \underline{\qquad\qquad})$

$A = -2(\underline{\qquad\qquad} - \underline{\qquad\qquad} + \underline{\qquad\qquad} - \underline{\qquad\qquad})$

$A = -2[(\underline{\qquad\qquad} - \underline{\qquad\qquad})^2 - \underline{\qquad\qquad}]$

$A = -2(\underline{\qquad\qquad} - \underline{\qquad\qquad})^2 + \underline{\qquad\qquad}$

The quadratic function representing all possible widths of Rylee's garden has its vertex at (3, 18). Rylee's garden will have a width of 3 m, a length of 6 m, and a maximum area of 18 m^2.

How is the length of 6 m determined? How do you know that the area is a maximum and not a minimum?

See pages 190–191 of *Pre-Calculus 11* for an example of optimization in a different context.

Check Your Understanding

Practise

1. Write each quadratic function in vertex form. State the coordinates of the vertex.

   a) $y = x^2 + 2x + 3$ b) $y = x^2 + 12x + 20$

   c) $y = -x^2 + 8x - 7$ d) $y = -x^2 - 10x - 31$

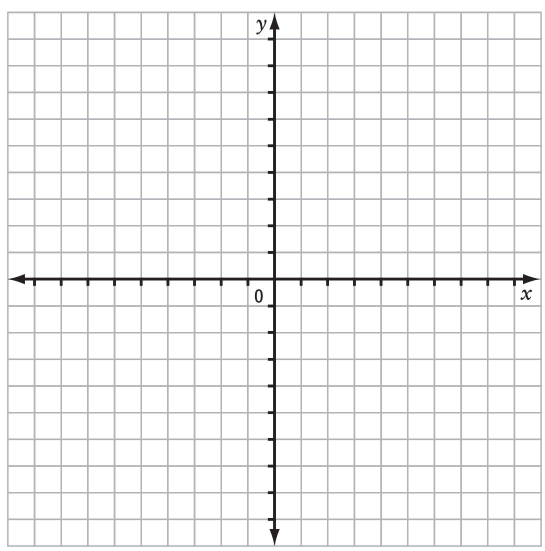
2. Write each function in vertex form by completing the square. State the coordinates of the vertex.

   a) $y = 2x^2 + 8x + 1$ b) $y = 5x^2 - 60x + 166$

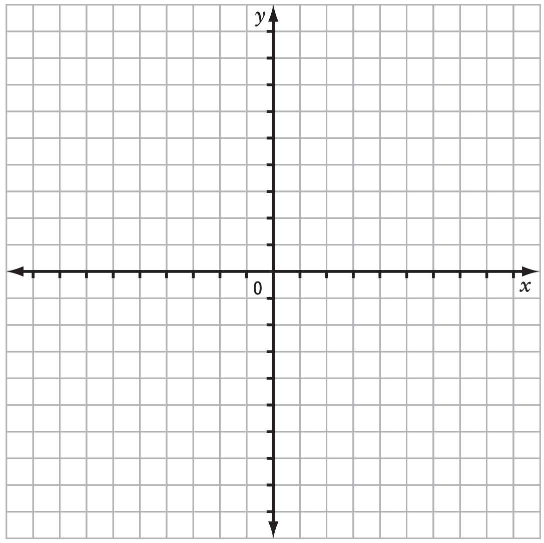
   c) $y = -4x^2 + 24x - 21$ d) $y = -7x^2 - 42x + 3$

3. Write each function in vertex form. Sketch the graph of the function, and label the vertex.

   a) $y = x^2 - 10x + 18$



b) $y = -2x^2 + 8x + 3$



4. Indicate which are quadratic functions. For the quadratic functions, expand and state the coordinates of the vertex.

a) $y = (x - 4)(x - 12)(x + 2)$

b) $y = 3(x + 8) - 2(x + 2)(x - 1) + 3x$

c) $y = 2(x - 6)(x - 3) - x + 5$

d) $y = (x - 3)(3x^2 + 6x - 1)$

5. Verify in two different ways that each pair of functions represents the same parabola.

a) $y = x^2 + 2x - 35$ and $y = (x + 1)^2 - 36$

b) $y = -2x^2 + 16x - 29$ and $y = -2(x - 4)^2 + 3$

c) $y = \frac{1}{2}(x - 5)^2 - 4$ and $y = \frac{1}{2}x^2 - 5x + \frac{17}{2}$

6. State the maximum or minimum value of each quadratic function, correct to the nearest hundredth, and the x-value at which it occurs.

a) $y = 3x^2 - 4x - 5$

b) $y = \frac{1}{3}x^2 - 4x + 10$

c) $y = -0.25x^2 + 2x + 3$

d) $y = 2x^2 - \frac{1}{4}x + 1$

See #7 and #8 on page 193 of *Pre-Calculus 11* for more practice in completing the square with fractions and decimals.

Apply

7. A manufacturer determines that the cost, c, in dollars, of producing a particular component can be modelled by the function $c(n) = 50n^2 - 8000n + 3\,300\,000$, where n is the number of components made. Determine the number of components that should be made so that the manufacturer has the minimum possible cost.

This quadratic function has a minimum because __________________________.
The minimum value occurs at the vertex of the function. So, determine the minimum by completing the square.

What information does the coordinates of the vertex give the manufacturer?

$c(n) = 50n^2 - 8000n + 3\,300\,000$

$c(n) = 50(n^2 - \_\_\_\_\_\_\_\_ n) + 3\,300\,000$

For more practice with this concept, complete #13 on page 194 of *Pre-Calculus 11*.

8. When an object is thrown in the air, its height, h, in metres after t seconds can be approximated by the function $h(t) = -5t^2 + vt + d$, where v is its starting speed in metres per second and d is its initial height, in metres, above the ground.

a) Suppose a ball is thrown from a height of 2 m at a speed of 20 m/s. Determine algebraically the maximum height of the ball and the time it takes to reach that height.

b) Use technology to verify your answer to part a).

c) A better model for the height of an object is $h(t) = -4.905t^2 + vt + d$. Use this model to determine the maximum height of this ball. Explain whether you prefer to use an algebraic or technological approach.

9. A sales manager wants to increase his sales revenue. Currently, he sells his product for $12 and sells 500 each month. His research indicates that for every $1 price increase, he will sell 25 fewer products.

a) Write a quadratic function that models this situation.

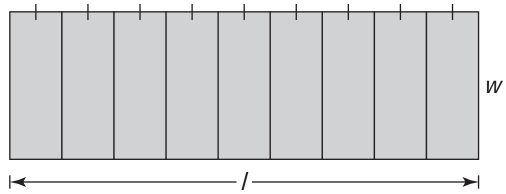
b) Rewrite the function in vertex form.

c) What does the vertex represent in this situation?

d) What price should the sales manager set for the product? How much revenue can he expect?

If you need help setting up the function to model this situation, see Example 4 on page 190 of *Pre-Calculus 11*.

10. An animal rescue society needs to build a new rectangular enclosure for 9 animals. The budget allows for the purchase of 100 m of fencing. The design of the enclosure is shown. What dimensions will provide the maximum area?



a) Write a function that represents the fencing used to build this enclosure.

b) Isolate one variable in your expression from part a).

c) Write a quadratic function that models the area of the enclosure.

d) Determine the vertex of the quadratic function you wrote in part c).

e) What dimensions give the maximum area for the enclosure?

Chapter 3 Review

3.1 Investigating Quadratic Functions in Vertex Form, pages 107–119

1. For each of the following, determine the number of x-intercepts, the equation of the axis of symmetry, and the domain and range.

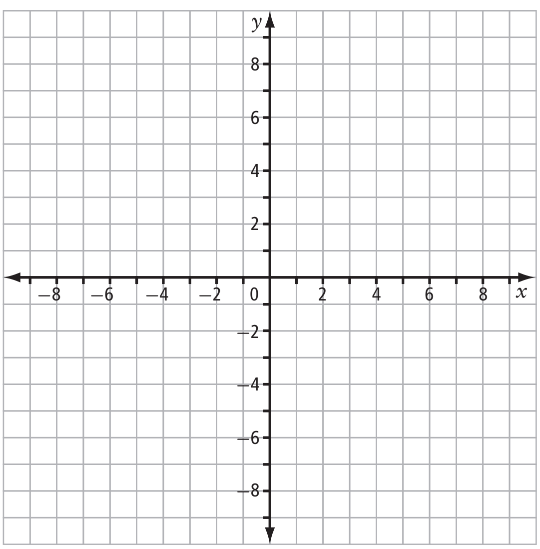
a) $y = -2(x + 5)^2 + 6$ **b)** $y = 5(x - 8)^2$

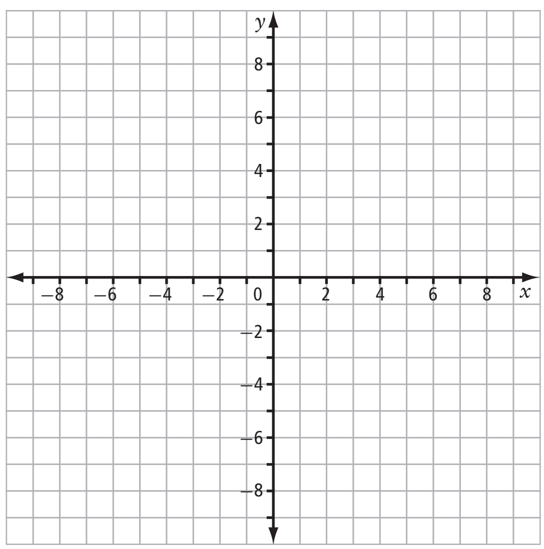
2. For each of the following, determine the coordinates of the vertex and whether the graph has a maximum or minimum value.

a) $y = -(x - 3)^2 - 7$ **b)** $y = 0.5(x + 11)^2 + 8$

3. Sketch each of the following functions. Label the vertex and axis of symmetry.

a) $y = -4(x + 1)^2$ **b)** $y = \frac{1}{4}(x + 2)^2 - 3$





4. Suppose a sculptor wants to create a parabolic arch with a height of 5 m and a width at the base of 8 m.

   a) Determine the quadratic function that represents the arch if the vertex of the parabola is at the origin.

   b) Determine the quadratic function that represents the arch if the origin is at the lower left end of the arch.

   c) Explain the similarities and differences between your two functions.

3.2 Investigating Quadratic Functions in Standard Form, pages 120–132

5. State the x-intercepts and y-intercept for each function.

   a) $y = x^2 + 2x - 8$

   b) $y = x^2 + 10x + 9$

6. Determine the x-coordinate of the vertex of each of the quadratic functions.

   a) $y = 2x^2 + 6x - 5$

   b) $y = -3x^2 - 5x + 9$

7. State the equation of the axis of symmetry and the direction of opening for each quadratic function.

   a) $y = -0.5x^2 - 5x + 2$

   b) $y = 6x^2 - 8x - 11$

3.3 Completing the Square, pages 133–141

8. Write each function in vertex form. State the domain and range.

a) $y = x^2 + 6x + 15$

b) $y = -3x^2 - 36x - 100$

c) $y = 2x^2 - 16x + 22$

d) $y = \frac{1}{2}x^2 - x + 3$

9. The profit, p, earned from the sale of a particular product by a business is given by $p(d) = -0.25d^2 + 5d + 80$, where d is the number of days the product has been for sale.

a) Determine the vertex of the profit function.

b) Explain what the vertex means in the context of this problem.

10. A student club is planning a fundraising car wash. Last year they charged \$10 per vehicle and washed 120 vehicles. They would like to earn more money this year. For every \$1 increase in price, they know they will wash 5 fewer vehicles.

a) Write a quadratic function to model this situation using v as the number of vehicles and r as the revenue.

b) Determine the best price to charge for the car wash and the revenue expected at that price.

Chapter 3 Skills Organizer

Complete the table for quadratic functions in vertex form, $f(x) = a(x - p)^2 + q$.

Parameter	Possible Value	Effect on Graph	Sketch
a	$a < -1$		
	$-1 < a < 0$		
	$0 < a < 1$		
	$a > 1$		
p	$p > 0$		
	$p < 0$		
q	$q > 0$		
	$q < 0$		

Complete the table for quadratic functions.

Equation	Coordinates of Vertex	Direction of Opening	y-Intercept
$y = ax^2 + bx + c$			
$y = a(x - p)^2 + q$			